COUPLED PAINLEVÉ SYSTEMS WITH AFFINE WEYL GROUP SYMMETRY OF TYPES $D_3^{(2)}$ AND $D_5^{(2)}$

YUSUKE SASANO

Abstract. In this paper, we find a two-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $D_3^{(2)}$. We also find a four-parameter family of 2-coupled $D_3^{(2)}$-systems in dimension eight with affine Weyl group symmetry of type $D_5^{(2)}$. We show that for each system, we give its symmetry and holomorphy conditions, respectively. These symmetries, holomorphy conditions and invariant divisors are new.

1. Introduction

In [1, 4, 6], we presented some types of coupled Painlevé systems with various affine Weyl group symmetries. In this paper, we find a 2-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $D_3^{(2)}$ explicitly given by

$$\frac{dq_1}{dt}=\frac{\partial H_{D_3^{(2)}}}{\partial p_1},\quad \frac{dp_1}{dt}=-\frac{\partial H_{D_3^{(2)}}}{\partial q_1},\quad \frac{dq_2}{dt}=\frac{\partial H_{D_3^{(2)}}}{\partial p_2},\quad \frac{dp_2}{dt}=-\frac{\partial H_{D_3^{(2)}}}{\partial q_2} \tag{1}$$

with the polynomial Hamiltonian

$$\begin{aligned} &H_{D_3^{(2)}}(q_1,p_1,q_2,p_2,t;\alpha_0,\alpha_1)\\ &=2H_{II}(q_1,p_1,t;\alpha_0)+H_{II}^{auto}(q_2,p_2;\alpha_1)+4p_1p_2-2q_1q_2p_2\\ &=2(q_1^2p_1+p_1^2+tp_1+\alpha_0q_1)+q_2^2p_2-2p_2^2+\alpha_1q_2+4p_1p_2-2q_1q_2p_2. \end{aligned} \tag{2}$$

Here q_1,p_1,q_2 and p_2 denote unknown complex variables, and $\alpha_0,\alpha_1,\alpha_2$ are complex parameters satisfying the relation:

$$\alpha_0+\alpha_1+\alpha_2=\frac{1}{2}. \tag{3}$$

The symbols H_{II} and H_{II}^{auto} denote

$$\begin{aligned} H_{II}(x,y,t;\alpha_0)=&x^2y+y^2+ty+\alpha_0x\\ H_{II}^{auto}(z,w;\alpha_1)=&z^2w-2w^2+\alpha_1z, \end{aligned} \tag{4}$$

where H_{II} denotes the second Painlevé Hamiltonian, and H_{II}^{auto} denotes an autonomous version of the second Painlevé Hamiltonian. Of course, the system with the Hamiltonian H_{II}^{auto} has itself as its first integral.

This is the first example which gave higher order Painlevé type systems of type $D_3^{(2)}$.

2000 *Mathematics Subjet Classification.* 34M55; 34M45; 58F05; 32S65.

Key words and phrases. Affine Weyl group, birational symmetry, coupled Painlevé system.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to q_1, p_1, q_2, p_2. However, we can not find. Of course, the Hamiltonian $H_{D_3^{(2)}}$ is not the first integral.

We also find a 4-parameter family of 2-coupled $D_3^{(2)}$-systems in dimension eight with affine Weyl group symmetry of type $D_5^{(2)}$ explicitly given by

$$\frac{dq_1}{dt} = \frac{\partial H_{D_5^{(2)}}}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial H_{D_5^{(2)}}}{\partial q_1}, \dots, \frac{dq_4}{dt} = \frac{\partial H_{D_5^{(2)}}}{\partial p_4}, \quad \frac{dp_4}{dt} = -\frac{\partial H_{D_5^{(2)}}}{\partial q_4} \tag{5}$$

with the polynomial Hamiltonian

$$\begin{aligned} H_{D_5^{(2)}} =& H_{D_3^{(2)}}(q_1, p_1, q_2, p_2, t; \alpha_0, \alpha_1) + H_{D_3^{(2)}}(q_4, p_4, q_3, p_3, t; \alpha_4, \alpha_3) \\ & - \frac{3}{2}p_1^2 - \frac{3}{2}p_4^2 + 3p_1 p_4. \end{aligned} \tag{6}$$

Here $q_1, p_1, q_2, p_2, q_3, p_3, q_4$ and p_4 denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_4$ are complex parameters satisfying the relation:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1. \tag{7}$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to $q_1, p_1, \dots, q_4, p_4$. However, we can not find. Of course, the Hamiltonian $H_{D_5^{(2)}}$ is not the first integral.

This is the second example which gave higher order Painlevé type systems of type $D_5^{(2)}$.

We also remark that 2-coupled Painlevé III system in dimension four given in the paper [5] admits the affine Weyl group symmetry of type $D_5^{(2)}$ as the group of its Bäcklund transformations, whose generators s_1, s_2, s_3 are determined by the invariant divisors. However, the transformations s_0, s_4 do not satisfy so (see Theorem 4.1 in [5]).

On the other hand, the system (16) admits the affine Weyl group symmetry of type $D_5^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, \dots, s_4$ are determined by the invariant divisors (3.2) (see Section 3).

We show that for each system, we give its symmetry and holomorphy conditions, respectively. These Bäcklund transformations of each system satisfy

$$s_i(g) = g + \frac{\alpha_i}{f_i}\{f_i, g\} + \frac{1}{2!}\left(\frac{\alpha_i}{f_i}\right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[q_1, p_1, \dots, q_4, p_4]), \tag{8}$$

where poisson bracket $\{,\}$ satisfies the relations:

$$\{p_1, q_1\} = \{p_2, q_2\} = \{p_3, q_3\} = \{p_4, q_4\} = 1, \quad \textit{the others are } 0.$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

These symmetries, holomorphy conditions and invariant divisors are new.

2. $D_3^{(2)}$ SYSTEM

In this paper, we study a 2-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $D_3^{(2)}$ explicitly given by

$$
\left\{
\begin{aligned}
\frac{dq_1}{dt} &= \frac{\partial H_{D_3^{(2)}}}{\partial p_1} = 2q_1^2+4p_1+2t+4p_2,\\
\frac{dp_1}{dt} &= -\frac{\partial H_{D_3^{(2)}}}{\partial q_1} = -4q_1p_1+2q_2p_2-2\alpha_0,\\
\frac{dq_2}{dt} &= \frac{\partial H_{D_3^{(2)}}}{\partial p_2} = q_2^2-4p_2+4p_1-2q_1q_2,\\
\frac{dp_2}{dt} &= -\frac{\partial H_{D_3^{(2)}}}{\partial q_2} = -2q_2p_2+2q_1p_2-\alpha_1
\end{aligned}
\right. \tag{9}
$$

with the polynomial Hamiltonian (2).

Theorem 2.1. *This system* (9) *admits the affine Weyl group symmetry of type* $D_3^{(2)}$ *as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation* $(*) := (q_1,p_1,q_2,p_2,t;\alpha_0,\alpha_1,\alpha_2)$,

$$
\begin{aligned}
s_0 : (*) &\rightarrow \left(q_1+\frac{4\alpha_0}{4p_1+q_2^2}, p_1, q_2, p_2-\frac{2\alpha_0 q_2}{4p_1+q_2^2}, t; -\alpha_0, \alpha_1+2\alpha_0, \alpha_2\right),\\
s_1 : (*) &\rightarrow \left(q_1, p_1, q_2+\frac{\alpha_1}{p_2}, p_2, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1\right),\\
s_2 : (*) &\rightarrow (q_1+\frac{4\alpha_2}{4p_1+8p_2+4q_1q_2-q_2^2+4t}, p_1-\frac{4\alpha_2 q_2}{4p_1+8p_2+4q_1q_2-q_2^2+4t}\\
&\quad -\frac{16{\alpha_2}^2}{(4p_1+8p_2+4q_1q_2-q_2^2+4t)^2}, q_2+\frac{8\alpha_2}{4p_1+8p_2+4q_1q_2-q_2^2+4t},\\
&\quad p_2-\frac{2\alpha_2(2q_1-q_2)}{4p_1+8p_2+4q_1q_2-q_2^2+4t}, t; \alpha_0, \alpha_1+2\alpha_2, -\alpha_2).
\end{aligned} \tag{10}
$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

Proposition 2.2. *This system has the following invariant divisors*:

parameter's relation	f_i
$\alpha_0=0$	$f_0 := p_1+\frac{q_2^2}{4}$
$\alpha_1=0$	$f_1 := p_2$
$\alpha_2=0$	$f_2 := p_1+2p_2+t+q_1q_2-\frac{q_2^2}{4}$

We note that when $\alpha_1=0$, we see that the system (9) admits a particular solution $p_2=0$. The system in the variables q_1,p_1 and q_2 are given by

$$\left\{\begin{aligned} \frac{dq_1}{dt}&=2q_1^2+4p_1+2t,\\ \frac{dp_1}{dt}&=-4q_1p_1-2\alpha_0,\\ \frac{dq_2}{dt}&=q_2^2+4p_1-2q_1q_2. \end{aligned}\right. \tag{11}$$

This is a Riccati extension of the second Painlevé system in the variables (q_1,p_1). Moreover, $\alpha_0=0$, we see that the system (11) admits a particular solution $p_1=0$. The system in the variables q_1 and q_2 are given by

$$\left\{\begin{aligned} \frac{dq_1}{dt}&=2q_1^2+2t,\\ \frac{dq_2}{dt}&=q_2^2-2q_1q_2. \end{aligned}\right. \tag{12}$$

This is a Riccati extension of Airy equation in the variable q_1.

When $\alpha_2=0$, after we make the birational and symplectic transformations:

$$x_2=q_1,\ y_2=p_1+2p_2+t+q_1q_2-\frac{q_2^2}{4},\ z_2=q_2-2q_1,\ w_2=p_2+\frac{q_1^2}{2}-\frac{q_2^2}{8} \tag{13}$$

we see that the system (9) admits a particular solution $y_2=0$.

PROPOSITION 2.3. *Let us define the following translation operators*:

$$T_1:=s_1s_2s_1s_0, \quad T_2:=s_1s_0s_1s_2. \tag{14}$$

These translation operators act on parameters α_i as follows:

$$\begin{aligned} T_1(\alpha_0,\alpha_1,\alpha_2)=&(\alpha_0,\alpha_1,\alpha_2)+(-1,1,0),\\ T_2(\alpha_0,\alpha_1,\alpha_2)=&(\alpha_0,\alpha_1,\alpha_2)+(0,1,-1). \end{aligned} \tag{15}$$

THEOREM 2.4. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H\in\mathbb{C}(t)[q_1,p_1,q_2,p_2]$. We assume that*

(A1) $deg(H)=3$ with respect to q_1,p_1,q_2,p_2.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system r_i $(i=0,1,2)$:

$$\begin{aligned} r_0:&x_0=\frac{1}{q_1},\ y_0=-\left(\left(p_1+\frac{q_2^2}{4}\right)q_1+\alpha_0\right)q_1,\ z_0=q_2,\ w_0=p_2+\frac{q_1q_2}{2},\\ r_1:&x_1=q_1,\ y_1=p_1,\ z_1=\frac{1}{q_2},\ w_1=-(q_2p_2+\alpha_1)q_2,\\ r_2:&x_2=\frac{1}{q_1},\ y_2=-\left(\left(p_1+2p_2+t+q_1q_2-\frac{q_2^2}{4}\right)q_1+\alpha_2\right)q_1,\\ &z_2=q_2-2q_1,\ w_2=p_2+\frac{q_1^2}{2}-\frac{q_2^2}{8}. \end{aligned}$$

Then such a system coincides with this system (9) *with the polynomial Hamiltonian* (2).

By this theorem, we can also recover the parameter's relation (3).

We note that the condition $(A2)$ should be read that

$$r_j(K) \quad (j=0,1), \quad r_2(K-q_1)$$

are polynomials with respect to x_i, y_i, z_i, w_i.

3. $D_5^{(2)}$ SYSTEM

We study a 4-parameter family of 2-coupled $D_3^{(2)}$-systems in dimension eight with affine Weyl group symmetry of type $D_5^{(2)}$ explicitly given by

$$\left\{\begin{aligned}
\frac{dq_1}{dt} &= \frac{\partial H_{D_5^{(2)}}}{\partial p_1} = 2q_1^2+p_1+2t+4p_2+3p_4,\\
\frac{dp_1}{dt} &= -\frac{\partial H_{D_5^{(2)}}}{\partial q_1} = -4q_1p_1+2q_2p_2-2\alpha_0,\\
\frac{dq_2}{dt} &= \frac{\partial H_{D_5^{(2)}}}{\partial p_2} = q_2^2-4p_2+4p_1-2q_1q_2,\\
\frac{dp_2}{dt} &= -\frac{\partial H_{D_5^{(2)}}}{\partial q_2} = -2q_2p_2+2q_1p_2-\alpha_1,\\
\frac{dq_3}{dt} &= \frac{\partial H_{D_5^{(2)}}}{\partial p_3} = q_3^2-4p_3+4p_4-2q_3q_4,\\
\frac{dp_3}{dt} &= -\frac{\partial H_{D_5^{(2)}}}{\partial q_3} = -2q_3p_3+2q_4p_3-\alpha_3,\\
\frac{dq_4}{dt} &= \frac{\partial H_{D_5^{(2)}}}{\partial p_4} = 2q_4^2+p_4+2t+3p_1+4p_3,\\
\frac{dp_4}{dt} &= -\frac{\partial H_{D_5^{(2)}}}{\partial q_4} = -4q_4p_4+2q_3p_3-2\alpha_4
\end{aligned}\right. \tag{16}$$

with the polynomial Hamiltonian (6).

THEOREM 3.1. *This system* (16) *admits extended affine Weyl group symmetry of type* $D_5^{(2)}$ *as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation* $(*) := (q_1,p_1,\dots,q_4,p_4,,t;\alpha_0,\alpha_1,\dots,\alpha_4)$,

$$\begin{aligned}
s_0:(*)\rightarrow&(q_1+\frac{4\alpha_0}{4p_1+q_2^2},p_1,q_2,p_2-\frac{2\alpha_0q_2}{4p_1+q_2^2},q_3,p_3,q_4,p_4,t;\\
&-\alpha_0,\alpha_1+2\alpha_0,\alpha_2,\alpha_3,\alpha_4),\\
s_1:(*)\rightarrow&\left(q_1,p_1,q_2+\frac{\alpha_1}{p_2},p_2,q_3,p_3,q_4,p_4,t;\alpha_0+\alpha_1,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4\right),
\end{aligned} \tag{17}$$

$$
\begin{aligned}
s_2:(*)\rightarrow&(q_1+\frac{2\alpha_2}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&p_1-\frac{2\alpha_2q_2}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t}\\
&-\frac{4\alpha_2^2}{(2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t)^2},\\
&q_2+\frac{4\alpha_2}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&p_2-\frac{\alpha_2(2q_1-q_3)}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&q_3+\frac{4\alpha_2}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&p_3-\frac{\alpha_2(2q_4-q_2)}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&q_4+\frac{2\alpha_2}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t},\\
&p_4-\frac{2\alpha_2q_3}{2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t}\\
&-\frac{4\alpha_2^2}{(2p_1+4p_2+4p_3+2p_4+2q_1q_2-q_2q_3+2q_3q_4+4t)^2},\\
&t;\alpha_0,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4),\\
s_3:(*)\rightarrow&\left(q_1,p_1,q_2,p_2,q_3,p_3+\frac{\alpha_3}{p_3},q_4,p_4,t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3\right),\\
s_4:(*)\rightarrow&(q_1,p_1,q_2,p_2,q_3,p_3-\frac{2\alpha_4q_3}{4p_4+q_3^2},q_4,p_4+\frac{4\alpha_4}{4p_4+q_3^2},t;\\
&\alpha_0,\alpha_1,\alpha_2,\alpha_3+2\alpha_4,-\alpha_4),\\
\pi:(*)\rightarrow&(q_4,p_4,q_3,p_3,q_2,p_2,q_1,p_1,t;\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0).
\end{aligned}
$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

PROPOSITION 3.2. *This system has the following invariant divisors*:

parameter's relation	f_i
$\alpha_0=0$	$f_0:=p_1+\frac{q_2^2}{4}$
$\alpha_1=0$	$f_1:=p_2$
$\alpha_2=0$	$f_2:=p_2+\frac{p_1+p_4}{2}+p_3+t+\frac{q_1q_2}{2}+\frac{q_3q_4}{2}-\frac{q_2q_3}{4}$
$\alpha_3=0$	$f_3:=p_3$
$\alpha_4=0$	$f_4:=p_4+\frac{q_3^2}{4}$

Proposition 3.3. *Let us define the following translation operators*:

$$T_1 := s_1s_2s_3s_4s_3s_2s_1s_0, \quad T_2 := s_1T_1s_1, \quad T_3 := s_2T_2s_2, \quad T_4 := s_3T_3s_3. \tag{18}$$

These translation operators act on parameters α_i as follows:

$$\begin{aligned} T_1(\alpha_0,\alpha_1,\dots,\alpha_4) =&(\alpha_0,\alpha_1,\dots,\alpha_4)+(-2,2,0,0,0),\\ T_2(\alpha_0,\alpha_1,\dots,\alpha_4) =&(\alpha_0,\alpha_1,\dots,\alpha_4)+(0,-2,2,0,0),\\ T_3(\alpha_0,\alpha_1,\dots,\alpha_4) =&(\alpha_0,\alpha_1,\dots,\alpha_4)+(0,0,-2,2,0),\\ T_4(\alpha_0,\alpha_1,\dots,\alpha_4) =&(\alpha_0,\alpha_1,\dots,\alpha_4)+(0,0,0,-2,2). \end{aligned} \tag{19}$$

Theorem 3.4. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[q_1,p_1,\dots,q_4,p_4]$. We assume that*

$(B1)$ *$deg(H)=3$ with respect to $q_1,p_1,\dots,q_4,p_4$.*

$(B2)$ *This system becomes again a polynomial Hamiltonian system in each coordinate system r_i $(i=0,1,\dots,4)$:*

$$\begin{aligned} r_0 :&x_0=\frac{1}{q_1},\ y_0=-\left(\left(p_1+\frac{q_2^2}{4}\right)q_1+\alpha_0\right)q_1,\ z_0=q_2,\ w_0=p_2+\frac{q_1q_2}{2},\\ &l_0=q_3,\ m_0=p_3,\ n_0=q_4,\ u_0=p_4,\\ r_1 :&x_1=q_1,\ y_1=p_1,\ z_1=\frac{1}{q_2},\ w_1=-(q_2p_2+\alpha_1)q_2,\\ &l_1=q_3,\ m_1=p_3,\ n_1=q_4,\ u_1=p_4,\\ r_2 :&x_2=q_1-\frac{q_2}{2},\ y_2=p_1+\frac{q_2^2}{4},\ z_2=\frac{1}{q_2},\\ &w_2=-\left(\left(p_2+\frac{p_1+p_4}{2}+p_3+t+\frac{q_1q_2}{2}+\frac{q_3q_4}{2}-\frac{q_2q_3}{4}\right)q_2+\alpha_2\right)q_2,\\ &l_2=q_3-q_2,\ m_2=p_3-\frac{q_2^2}{4}+\frac{q_2q_4}{2},\ n_2=q_4-\frac{q_2}{2},\ u_2=p_4+\frac{q_2q_3}{2}-\frac{q_2^2}{4},\\ r_3 :&x_3=q_1,\ y_3=p_1,\ z_3=q_2,\ w_3=p_2,\ l_3=\frac{1}{q_3},\ m_3=-(q_3p_3+\alpha_3)q_3,\\ &n_3=q_4,\ u_3=p_4,\\ r_4 :&x_4=q_1,\ y_4=p_1,\ z_4=q_2,\ w_4=p_2,\ l_4=q_3,\ m_4=p_3+\frac{q_3q_4}{2},\ n_4=\frac{1}{q_4},\\ &u_4=-\left(\left(p_4+\frac{q_3^2}{4}\right)q_4+\alpha_4\right)q_4. \end{aligned}$$

Then such a system coincides with this system (16) *with the polynomial Hamiltonian* (6)*.*

By this theorem, we can also recover the parameter's relation (7).

We note that the condition $(B2)$ should be read that

$$r_j(K) \quad (j=0,1,3,4), \quad r_2(K-q_2)$$

are polynomials with respect to $x_i,y_i,z_i,w_i,l_i,m_i,n_i,u_i$.

References

[1] Y. Sasano, *Higher order Painlevé equations of type* ${D_l}^{(1)}$, RIMS Kokyuroku **1473** (2006), 143–163.

[2] Y. Sasano, *Coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of types* $B_6^{(1)}, D_6^{(1)}$ *and* $D_7^{(2)}$, preprint.

[3] Y. Sasano, *Coupled Painlevé II systems in dimension four and the systems of type* $A_4^{(1)}$, Tohoku Math. **58** (2006), 529–548.

[4] Y. Sasano, *Symmetry in the Painlevé systems and their extensions to four-dimensional systems*, Funkcial. Ekvac. **51** (2008), 351–369.

[5] Y. Sasano, *Coupled Painlevé III systems with affine Weyl group symmetry of types* $B_4^{(1)}$*,* $D_4^{(1)}$ *and* $D_5^{(2)}$, preprint.

[6] Y. Sasano, *Coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type* $D_6^{(1)}$*, II*, RIMS Kokyuroku Bessatsu. **B5** (2008), 137–152.

[7] C. M. Cosgrove, *Higher order Painlevé equations in the polynomial class II, Bureau symbol P1*, Studies in Applied Mathematics. **116** (2006).

[8] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Société Mathématique de France. **28** (1900), 201–261.

[9] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale est uniforme*, Acta Math. **25** (1902), 1–85.

[10] B. Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes*, Acta Math. **33** (1910), 1–55.

[11] C. M. Cosgrove and G. Scoufis, *Painlevé classification of a class of differential equations of the second order and second degree*, Studies in Applied Mathematics. **88** (1993), 25-87.

[12] C. M. Cosgrove, *All binomial-type Painlevé equations of the second order and degree three or higher*, Studies in Applied Mathematics. **90** (1993), 119-187.

[13] F. Bureau, *Integration of some nonlinear systems of ordinary differential equations*, Annali di Matematica. **94** (1972), 345–359.

[14] J. Chazy, *Sur les équations différentielles dont l'intégrale générale est uniforme et admet des singularités essentielles mobiles*, Comptes Rendus de l'Académie des Sciences, Paris. **149** (1909), 563–565.

[15] J. Chazy, *Sur les équations différentielles dont l'intégrale générale posséde une coupure essentielle mobile* , Comptes Rendus de l'Académie des Sciences, Paris. **150** (1910), 456–458.

[16] J. Chazy, *Sur les équations différentielles du trousiéme ordre et d'ordre supérieur dont l'intégrale a ses points critiques fixes*, Acta Math. **34** (1911), 317–385.